Numerical Modeling of Passive Microwave Devices

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(Invited Paper)

***Abstract*—Passive linear bilateral microwave devices principally include guided wave transmission structures, their junctions, and interconnections, as well as such complex devices as resonant cavities and planar networks. The computational methods for analyzing the behavior of this class of structures and for characterizing them for purposes of network design are briefly reviewed.**

I. Introduction

A PASSIVE microwave device may be defined as a closed box filled with a heterogeneous combination of passive material media, connected to the rest of the universe through a finite number of clearly definable ports. This definition contains within it the classical and general definition of a waveguide junction, if all the ports are of the closed waveguide type. The object of the present paper is to review recent literature in the area of numerical modeling of passive microwave devices and to provide a selected bibliography of contributions to this field.

It may be noted that the literature available on passive microwave devices is extremely extensive and even when the discussion is restricted to the computational methods used in modeling passive devices, an enormous amount of material remains. To keep the subject matter within reasonable bounds, we have endeavored to choose a representative cross section of the available literature; no attempt has been made to create an exhaustive bibliography. Indeed, it is doubtful whether a complete bibliography for so broad a field would be useful. Our "raw" bibliography compiled while writing this paper included over 1000 items. A rough estimate of the total available literature may be obtained by checking the raw bibliography against citations in recently published papers. From such internal evidence, it would appear that the world literature on passive device modeling presently comprises about 10 000 papers, of which about half may be termed "recent." Since our interest in this paper is primarily expository rather than historical, we shall concentrate heavily on recent papers in the field, with only an occasional reference to work done a decade or more ago. Furthermore, we recognize that the distinction between numerical and analytic methods is sometimes blurred, and will use as a guide in the survey the criterion that a numerical method leads to an algorithm for the solution of a problem, whereas an analytic method yields an expression of more or less closed form.

In general, a microwave network may be regarded as a collection of active and passive devices—that is to say, finite-sized noninteracting blocks or boxes—whose ports are interconnected in some prescribed fashion by wave guides.[1] These may be standard hollow waveguides, transmission lines, dielectric surface guides, or any other wave-guiding structures. For present purposes, microwave network analysis may be regarded as a process in which each device and each section of wave guide is mathematically modeled by a matrix, the individual matrices being combined in a manner corresponding to the interconnection of guides and devices so as to arrive at a composite mathematical model for the entire network. In the following, we shall restrict our attention to interconnections of *passive bilateral linear* devices: there is a certain amount of unity in the mathematics used for their treatment, while devices including anisotropic or nonlinear materials (e.g., plasmas and ferrites) often require specialized methods.

The modeling procedure is a two-part process. First, the fields inside the device are approximated by linear combinations of functions chosen in advance; subsequently, the expansion coefficients of these approximating functions are related to port parameters. Once the approximating functions have been chosen, all matrix representations based on the same approximating functions can be obtained from each other by simple matrix transformations. For example, impedance matrices may be transformed into admittance matrices by inversion, or into the *ABCD* parameters of a two-port by a rearrangement of matrix entries. Which choice of representation is best is a decision for the microwave network analysis program designer: for modeling purposes, it is the choice of approximating functions that matters.

To illustrate the process by a simple physical example, consider the semi-infinite coaxial line terminated in an open circuit, shown in Fig. 1(a). It will be assumed that the operating frequency is low enough to allow only the TEM wave to propagate. A simple equivalent circuit, valid at low frequencies, consists of a semi-infinite length of "perfect" TEM line, with a terminating capacitor to account for the extra energy stored in the end region shown in Fig. 1(b). The essence of the passive device modeling problem is to find the correct value for this capacitor. It should be noted that this value and the length to be assigned to the transmission-line section both depend on

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[1] We distinguish in this paper between *wave guides* (longitudinally uniform structures capable of guiding waves) and *waveguides* (uniform closed metallic pipes).

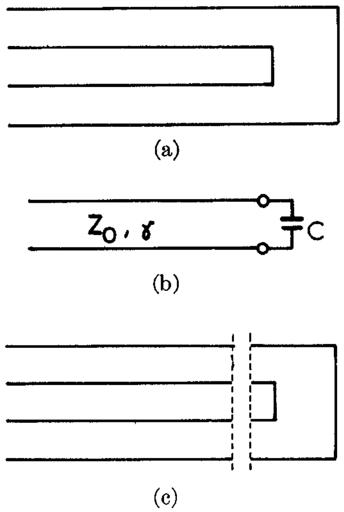


Fig. 1. (a) Semi-infinite coaxial line terminated in open circuit. (b) Low-frequency model, comprising idealized TEM line and lumped capacitor. (c) Boundary value problem for determining end capacitance.

where the imaginary division between transmission line and passive device (the so-called *terminal plane*) is made. It will be readily appreciated that the terminal plane location is entirely arbitrary, since the plane is wholly imaginary.

The equivalent network model of Fig. 1(b) postulates in effect that a pure unmodified TEM wave pair (one incident and one reflected wave) exists in the semi-infinite line, up to the terminal plane; one imagines that this wave pair suddenly disappears at the terminal plane and does not exist on its right-hand side. To the right of the terminal plane, a field is supposed to exist that satisfies Laplace's equation, subject to the given boundary conditions on all metal surfaces and to appropriate continuity conditions on the terminal plane. As a first approximation, one may imagine that the equipotential lines in this problem are essentially normal to the terminal plane everywhere. In that case, it suffices to solve the boundary value problem shown in Fig. 1(c) to find the fields in the end region beyond the terminal plane. The capacitance value may then be calculated by evaluating the stored energy in the field.

Consider next the modified open-circuited coaxial line shown in Fig. 2(a). This problem is exactly similar to the previous one, except that the end region has been enlarged, forming a resonant cavity of significant size. Since a resonant cavity has an infinite number of resonant frequencies, the correct equivalent circuit, Fig. 2(b), should theoretically include an infinite number of branches. Of course in practice only a few modes of the cavity will be considered so that only a few resonant branches occur in the network model. Indeed, it should be noted that the model of Fig. 1(b) represents the low-frequency asymptotic behavior of the model of Fig. 2(b).

At relatively high frequencies, modes other than TEM may propagate along the coaxial line. These modes are, of course, linearly independent throughout the length of the coaxial guide, and may therefore be modeled by

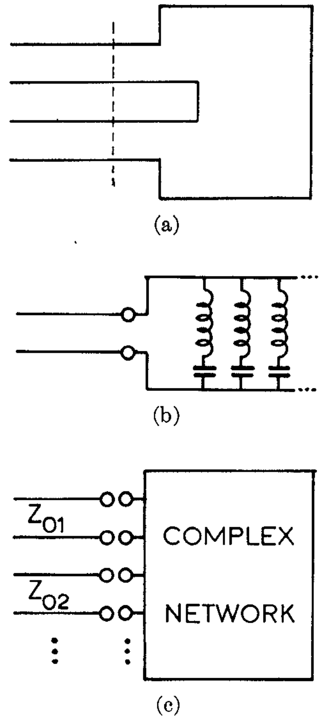


Fig. 2. (a) Coaxial line terminated in resonant cavity. (b) Network model valid at low frequencies. (c) Multiport model, valid at frequencies where multimode propagation can take place.

separate wave guides connected at the terminal plane.

If high-order modes in addition to the TEM mode propagate, the line-and-cavity problem of Fig. 2(a) requires the model shown in Fig. 2(c). The number of ports to be assigned to the network model clearly depends on the number of propagating modes that each transmission structure can support; one network port must be provided for each possible mode. For this reason, any model will remain valid only over a restricted frequency range.

The detailed mathematical techniques used in the modeling of passive devices vary enormously, yet the fundamental steps remain virtually the same. The first task of the analyst is to identify the possible propagating modes on each of the transmission structures at the frequency of interest. Secondly, terminal planes must be specified so as to define unambiguously which region of space will be considered to form the device to be modeled. Third, a choice must be made as to the description of the ports. Since a pair of waves will exist for each port, alternative representations are in terms of incident and reflected wave amplitudes or in terms of port voltage and current. The fourth step involves providing a description for all possible interior field configurations at the operating frequency; this is frequently done as an expansion in the normal modes of the device to be modeled, though often other representations are used. Fifth, the usual continuity requirements of electromagnetic theory are imposed at the ports so as to obtain relationships between the internal field and the amplitudes of the externally propagating incident and reflected waves.

In the following, the most common varieties of transmission structure are first considered, and methods for identifying their modal structure briefly reviewed: the TEM transmission line, the hollow waveguide, and microstrip. Thereafter, methods applicable to passive devices with one or more ports are examined.

II. Models of TEM Lines

TEM lines are by definition those wave guides in which only a single mode, the TEM mode, is assumed to propagate. This mode is completely described by a scalar potential ψ where

$$\psi = \phi(x,y) \cos (wt - kz) \tag{1}$$

the velocity of propagation being always equal to the velocity of light in the dielectric medium of the TEM line. The function ϕ in turn satisfied Laplace's equation

$$\nabla^2\phi(x,y) = 0 \tag{2}$$

in the plane transverse to the direction of propagation, and subject to the boundary conditions that ϕ must have exactly the same value everywhere on the surface of any single conductor forming part of the line.

The above two-dimensional boundary value problem, involving Laplace's equation with Dirichlet boundary conditions, is commonly used by numerical analysts as a model problem in the development of numerical methods. It is among the easiest of continuum problems to solve numerically, provided that the region of solution is finite; that is to say, provided that the TEM line is topologically of the coaxial type, in which one conductor forms an enclosing shield around the other. The classical method of finite differences is well suited to this problem and has been used, for example, by Metcalf [1] to determine the potential distribution and the characteristic impedance of lines composed of square coaxial conductors. The essence of the finite-difference method is to replace the continuum equation (2) by a large set of discrete algebraic equations that are solved iteratively. This class of techniques may be regarded as well established, and for details of implementation the reader is referred to Wexler's review paper [2] and to the book by Wachspress [3].

An alternative method that has proved popular, and of which many variants exist, requires the region in which Laplace's equation is to be solved to be divided into subregions, in each of which the equation may be solved by separation of variables. For example, Fig. 3 shows the cross-sectional shape of a curious coaxial line, which may be subdivided, as indicated, into an annular region and a rectangular region. In the rectangular region, solutions of Laplace's equation may be written as a double Fourier's series, whose coefficients are as yet unknown; in the annular region, a double series in Bessel functions and trigonometric functions serves the same purpose. Along the imaginary dividing line between subregions, the usual continuity equations of electromagnetics must hold. Hence the two infinite series expansions must have coefficients such that: a) Laplace's equation is satisfied in each

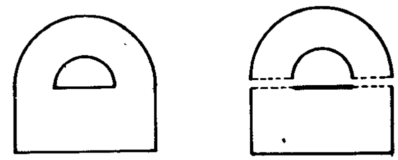


Fig. 3. Coaxial transmission-line problem permitting approximate solution by separation of variables in individual subregions, followed by imposition of continuity requirements along subregion boundaries.

subregion; and b) all fields are continuous across the dividing lines. These relationships between coefficients may be written in the form of infinite matrix equations, which may be solved numerically if each series is truncated to a finite number of terms. An early variant of this method was termed "orthonormal block analysis" by Cruzan and Garver [4]; numerous other versions have been published since then. Since it is obviously impossible to satisfy the continuity conditions everywhere along the dividing lines with only a finite number of adjustable coefficients, a variety of choices exists; one may require continuity at a selected number of points, or in a least squares sense at all points of the dividing line, or in various other ways, each of which effectively yields another distinct version of the general method.

The methods described above are strictly applicable to closed (i.e., coaxial) TEM lines. In the case of open lines, as for example the classical two-wire pair, considerable difficulties are encountered because it is not easy to model the boundary condition at infinity. There exists, of course, the possibility of introducing a purely artificial boundary at a location judged on physical grounds to be sufficiently far away. However, this approach leaves much to be desired both in terms of computational efficiency and in the amount of physical insight and judgment that must be used with the computer programs. In recent years, therefore, efforts have been made to develop methods that model the infinite exterior region more accurately. For example, Fig. 4 shows an open-wire pair composed of square wires; the region in which Laplace's equation must be satisfied is the entire x–y plane, except for the interiors of the wires themselves. An early method of solving this problem, termed "boundary relaxation" by Cermak and Silvester [5], proceeds to encase the wires in two nested window frames, as indicated in Fig. 4(a). If the values of potential everywhere on the outer frame A are assumed, there remains a straightforward boundary value problem that may be solved by finite-difference methods or by any of the other methods just discussed. Such a solution will yield, among other data, potential values on the inner window frame B. If potential values on this frame are known, however, potential values anywhere in the region exterior to B may be computed. This exterior solution will include within it new potential values on the frame A. It can be shown theoretically and has been demonstrated computationally that the potential values thus obtained on frame A are nearer the correct

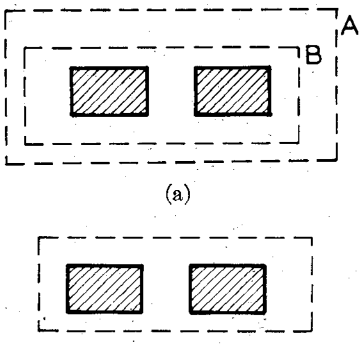


Fig. 4. (a) Two-frame formulation of open-boundaried problems. (b) Single frame may be used in alternative formulation.

values than the initial estimate; that is to say, the procedure described may be regarded as one step of a convergent iterative process. The end result of the process is a solution that looks as if the problem had been solved for the whole infinite plane, but only the portion enclosed within the window frame A was available for examination.

A second possibility, closely related to boundary relaxation, is to employ only a single frame, as shown in Fig. 4(b). The potential values anywhere in the exterior region, *including on the window frame itself*, satisfy the integral equation

$$\phi(x,y) = \oint G \left.\frac{\partial \phi}{\partial n}\right|_P dP \tag{3}$$

where G is the Green's function for the Laplacian operator and the integral extends over the frame contour. This integral equation may be approximated by a matrix equation. Similarly, the interior problem of Laplace's equation within the window frame of Fig. 4(b) may be approximated by a matrix equation. In this way, two matrix equations are set up that share the potential values along the window frame as unknowns. Simultaneous solution of both equations then yields potential values within and on the window frame. This method was proposed by Silvester and Hsieh [6], the interior discretization being accomplished by the finite-element method. Careful attention to the method of discretization is required in this technique, since the integral operator in (3) is invariably singular. A related method, employing two window frames as in Fig. 4(a), has recently been employed by McDonald and Wexler [7]; integration through the singularity is avoided in this way.

Once the potential distribution in a TEM line has been found, the characteristic impedance may be determined by finding the capacitance per unit length of line. A straightforward method for doing this is to find the normal derivative of the potential at the conductor surfaces (this is proportional to the charge density) and then to integrate over the conductors to find the total charge for the given potential difference. However, since numerical differentiation is very unstable, in most numerical methods of solution it is easier and more accurate to find the stored energy in the field and then to calculate the capacitance from the stored energy. A variationally stationary capacitance formulation suitable for use with finite-element methods based on energy has recently been given by Daly and Helps [8].

III. Hollow Waveguides

A hollow waveguide filled with homogeneous isotropic dielectric is capable of propagating two distinct sets of modes, TE and TM, and there exist infinitely many modes of each kind. The numerical problem of determining the possible propagating modes is therefore somewhat more complicated than the solution of a TEM line, where only one mode is possible. It can be shown that both types of modes satisfy the homogeneous Helmholtz equation

$$(\nabla^2 + k^2)\phi = 0 \tag{4}$$

subject to homogeneous Dirichlet or Neumann boundary conditions in the TM and TE cases, respectively. Equation (4) represents a classical eigenvalue problem, possessing infinitely many solutions corresponding to an infinity of eigenvalues k^2.

Finite-difference methods have been applied to the homogeneous waveguide problem. Davies and Muilwyk [9] reported what was probably the first general-purpose computer program using this method, although the technique itself had been known and extensively discussed much earlier. As in almost all numerical methods, with the finite-difference method the continuum equation (4) is replaced by the matrix equation

$$(A + k^2 B)\Phi = 0 \tag{5}$$

constructed in such a way that the eigenvalues and eigenvectors of the matrix problem give approximations to the eigenvalues and eigenvectors of the continuum problem. In this case, the matrix A has a particularly simple structure, while the matrix B is the identity matrix. However, the matrix order is typically several thousand, so that explicit methods are not very useful and iterative techniques are normally resorted to. An unfortunate difficulty with these iterative techniques is that regardless of the initial iterate, the iteration always converges to the lowest order propagating mode, and that high-order modes cannot be found. Matrix deflation (also called *mode subtraction*), which permits finding highest modes in general matrix problems, cannot be used with the sparse matrices encountered in finite-difference approximations without considerable difficulty. To circumvent this problem, Beaubien and Wexler [10] proposed an ingenious scheme. Once the dominant solution to (4) has been found, a new problem is constructed [11] in which the Laplacian operator in (4) is replaced by another operator possessing the same eigenvectors, but with the first eigenvalue removed. An iterative solution will then automatically converge to the second mode of (4). This process may be continued to determine as many propagating modes as desired, with the computing cost very roughly proportional to the number of modes calculated.

The method of subregions referred to above in connection with TEM lines, sometimes also called the *mode matching* method, has also found considerable application in the analysis of wave guides [12], [13]. The essence of the technique is very similar to that pursued in the solution of TEM line problems, except of course that a matrix eigenvalue problem results instead in an indeterminate set of simultaneous equations.

The direct Rayleigh–Ritz approximation method has been known for a long time, and it is surprising that it was not applied as a general procedure to waveguide problems until relatively recently. Its use is straightforward for the determination of TE modes [14], where the boundary conditions of the Rayleigh–Ritz functional are natural. Determination of the TM modes, however, for which the usual electromagnetic boundary conditions are principal and must be imposed upon the approximating functions from the start, are more difficult. For a large variety of waveguide shapes, however, construction of such functions is possible, making the Rayleigh–Ritz method quite useful. Approximating functions are typically chosen to be either polynomials in Cartesian coordinates or in polar coordinates [15]; in the former case, some difficulty may be encountered with reentrant cross sections, while the second choice is best if the cross sections are star shaped with respect to a point.

For most elliptic boundary value problems, an equivalent formulation in terms of a Fredholm integral equation of the second kind may be given, and the Helmholtz equation (4) is no exception. The possibility has certain attractions, since a two-dimensional boundary value problem is thus replaced by an integral equation involving a contour integral, which is computationally very economical. The disadvantage of this approach is that the linear eigenvalues appear in some complicated functional form. Two distinct formulations have been given by Spielman and Harrington [16] and by Ng and Bates [17]. In both cases, search techniques must be resorted to in order to determine the eigenvalues, although the matrix sizes are relatively small. As in any discrete search, there invariably lurks the danger of some eigenvalues being missed, particularly if they are nearly degenerate.

In the last few years, the finite-element method has gained considerable ground in the solution of boundary value problems and is at the present time preeminent in the solution of waveguide problems. The method may be loosely regarded as a combination of the method of subregions and the Ritz minimization method; it differs, however, fundamentally from the method of subregions in that the approximating functions employed are required to be continuous across subregion boundaries, but not necessarily to satisfy the differential equations exactly anywhere. The advantage of the method is that it combines the high accuracy and short computing times of the Rayleigh–Ritz method with the flexibility and ease of usage of the finite-difference method.

The first general-purpose finite-element computer program was described by Silvester [18], who used triangular elements and polynomial approximating functions of up to the fourth order. A much more extensive and flexible program, applicable to TEM lines as well as waveguides, and employing approximations up to sixth order was subsequently published by Konrad and Silvester [19]. These computer programs produce, with a minimum of data preparation, the solutions of arbitrary polygonal waveguides, generally with accuracies of better than 0.1 percent for the dominant and first few higher order wave numbers.

The solution of some specific waveguide problems with the finite-element method has been published by Lagasse and Van Bladel [20]. Furthermore, Daly [21] has recently described quasi-triangular elements with curved sides, which share many of the desirable properties of ordinary triangular elements. Isoparametric elements, long used in many areas of continuum analysis, have recently made an appearance in electromagnetics as well. It is to be expected, if the pattern of numerical analysis in other branches of engineering provides any indication, that the use of the finite-element method will become increasingly important to microwave engineers in future years.

IV. Microstrip and Related Guides

Strip, slot, and similar wave guides lend themselves well to microwave integrated circuit design. They have consequently enjoyed considerable popularity in the past decade.

Initial attempts to analyze microstrip relied on a TEM approximation and addressed themselves to the electrostatic problem of a charged microstrip [22], [23]. The results of such analyses, which have in the meantime been extended to multiple strips and multiple substrates, are accurate at very low frequencies where the propagating wave is essentially TEM in nature. In reality, however, hybrid waves are the only ones possible on an inhomogeneous-dielectric structure, and at frequencies well within the current range of technological interest, dispersion makes itself felt strongly enough to have a substantial effect on design. To date, no attempt has been published to extend the integral equation analysis of the electrostatic line to the dispersive case. An interesting ad hoc approximation has been published by Jain *et al.* [24], who attempted to correct the TEM analysis by adding a single surface-wave component. Another possibility is to transform the relevant field equations into the spectral domain, as reported by Denlinger [25] and by Itoh and Mittra [26], and to determine approximate solutions of the transformed equations. Virtually all other analyses, beginning with the first one published by Zysman and Varon [27], have addressed the related problem of a microstrip line enclosed by a shielding box, which can alternatively be regarded as the problem of a rectangular dielectric-slab loaded waveguide, with a metallic strip placed upon the dielectric slab.

For the shielded microstrip, several solutions are available. A finite-difference formulation was given by Hornsby and Gopinath [28]. Subsequently, Daly [29] used a finite-element formulation similar to that of Csendes and Silvester [30]. More recently, Corr and Davies [31] returned to the finite-difference formulation, using direct Householder transformations of the resulting matrices so as to avoid iterative methods whose convergence has long been known to the uncertain in this application [32]. Each of these investigators formulated the problem in terms of two Helmholtz equations (one for the longitudinal magnetic, the other for the electric field) coupled by the appropriate boundary conditions at the interface between dielectric media. The resulting solutions include not only the possible propagating modes, but also a number of other solutions that have no physical existence. The nonphysical solutions seem to result from choosing approximating functions out of function spaces that include function pairs not satisfying Maxwell's equations.

Since the shielded microstrip problem inherently involves rectangular geometries, it is a natural candidate for the method of subregions. Results obtained using this method have recently been published by Krage and Haddad [33] and by Kowalski and Pregla [34], among others.

Although several methods now exist for producing the current dispersion curve of the fundamental mode in microstrip, all of the analyses presented so far have the fundamental shortcoming that they do not automatically produce the mode patterns for the first few modes in a computationally convenient form. In addition to the difficulty alluded to above, all formulations that involve a shielding box necessarily add waveguide-like modes that may physically propagate in the real model, but which do not exist on unshielded microstrip. Thus the use of these results in circuit modeling still requires considerable manual intervention by skilled hands. In the authors' opinion, the definitive method for open microstrip is still to be published.

V. Waveguide Discontinuities

The numerical methods appearing in recent years on the analysis of waveguide discontinuities may be classified into four broad categories: modal approximation techniques; integral equation formulations; transverse discontinuity representations; and longitudinal discontinuity representations. Each of these types of procedures is based on a different technique for reducing waveguide discontinuity problems to one or two dimensions. It is perhaps indicative of the state of the art that for waveguide discontinuity problems, not a single truly three-dimensional solution has been published, although every waveguide discontinuity problem is inherently three dimensional in nature.

The geometry with which we will be concerned is indicated in Fig. 5, where the discontinuity is entirely confined between the planes a and b and the guiding structures are assumed to extend uniformly to infinity. The obstacle may be either metal or dielectric and is very often confined to either of the planes a or b. The solution of discontinuity problems involving the junction of three or more waveguides is obtained in essentially the same manner as the two waveguide case and will not be treated separately here.

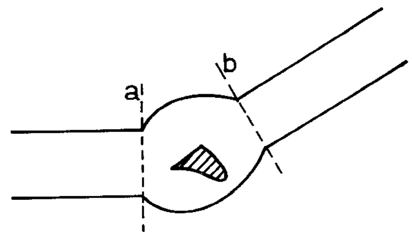


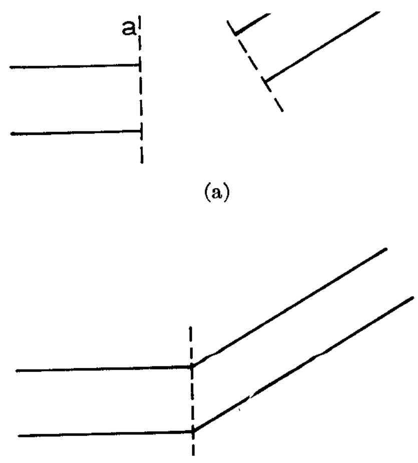
Fig. 5. Typical two-port passive device or wave guide discontinuity, consisting of a cavity enclosing an obstacle.

The mathematical statement of the problem depicted in Fig. 5 is that the electric and magnetic fields must satisfy the *vector* Helmholtz equation

$$(\nabla^2 + k^2)\phi = 0 \qquad \phi = \bar{E} \text{ or } \bar{H} \tag{6}$$

subject to the usual boundary conditions. As usual, the first step in determining a network model for this problem is to establish a hypothetical guiding structure having known and compatible physical characteristics. In the present case, this hypothetical guiding structure may be taken to be the two semi-infinite waveguide sections in Fig. 6. Using the methods of Section III, the solution of the two semi-infinite waveguides in Fig. 6(a) can be determined. This solution is then subtracted from the solution of the problem in Fig. 5, leaving only evanescent fields in the neighborhood of the discontinuity. The impedance matrix and the scattering parameters of the discontinuity are then calculated from the energy associated with this residue. The following sections will indicate the established procedures for solving the boundary value problem in Fig. 5.

As stated in the Introduction, the locations of the terminal planes are arbitrary. We could equally well have chosen the hypothetical guiding structure in Fig. 6(b)



(a)

(b)

Fig. 6. Two possible reference solutions for discontinuity problem of Fig. 5.

for the field reference, this structure being of the same overall length as the waveguide being modeled. The evanescent residue obtained by subtracting the fields in Fig. 6(b) from the fields in Fig. 5 is obviously different from that obtained by subtracting the fields in Fig. 6(b) from those in Fig. 5 and will result in a different characterization of the discontinuity. Thus values of negative capacitance and negative inductance can and often are obtained for microwave discontinuities; no physical contradiction arises, since these capacitances and inductances are not absolute quantities, only values relative to some hypothetical guiding structure.

VI. Modal Analysis

One of the best established numerical procedures for the solution of the boundary value problem defined by (6) is the modal analysis technique. This procedure is directly applicable to structures containing planar discontinuities only, although several discontinuity planes may appear in one problem. The idea behind the method is an extension of the discussion given above for the replacement of a waveguide discontinuity problem by two corresponding hypothetical semi-infinite uniform waveguides. Provided that the discontinuity is planar, as in the example in Fig. 7(a), a hypothetical wave-guiding structure may be defined with a region of definition exactly corresponding to the region defined by the original problem, as illustrated in Fig. 7(b). Now, it is well known that the modes in a uniform waveguide form a complete set of functions. Therefore, the modes corresponding to the hypothetical waveguide structure may be used as basis functions to approximate the electromagnetic fields in waveguides containing planar discontinuities. In fact, by using waveguide modes as approximation functions, the Helmholtz equation (6) and the necessary boundary conditions are satisfied everywhere except on the discontinuity plane P. Therefore, it is only necessary to enforce that the electric and magnetic fields satisfy the appropriate boundary conditions on the discontinuity plane P.

In the early use of the modal analysis technique, the equality of the electric and magnetic fields on the two sides of the plane was accomplished either by point matching or by matching Fourier coefficients. Examples of these types of analyses are the point matching and modal approach used by Shockley *et al.* [35] to solve a linearly tapered waveguide section problem and the Fourier coefficient matching analysis of Lucas [36] regarding waveguide planar offset problems. Experience indicates, however, that both of these approaches can lead to divergent results and therefore better methods of equating fields across the discontinuity plane are required. One successful approach to modal analysis was proposed independently in three separate papers in 1967, by Clarricoats and Slinn [37], Wexler [38], and Cole *et al.* [39]. In these papers, the electric and magnetic fields are made to satisfy the orthogonality condition

$$\int \bar{E} \times \bar{H} \cdot d\bar{S} = 0 \tag{7}$$

on the metal wall at P and across the interface between waveguides. This corresponds to the requirement that no net power may be transferred into metal walls and that the power leaving one guide equals the power entering the other.

Quite a number of papers have appeared in the literature since 1967 using the approach to solve particular waveguide discontinuity problems. In them, Bates [40] used modal analysis to determine the behavior of junctions between straight and curved waveguides; Huckle and Masterman [41] treated the H-plane junction of a waveguide containing a row of rectangular ports; Schlude [42] solved waveguide discontinuity problems in which thin irises occur; Krohne [43] studied the junction of two circular waveguides, one of which contained a coaxial dielectric rod; and McDonald [44] analyzed two identical resonant cavities coupled by an aperature of arbitrary thickness.

In addition to these papers, one further paper has appeared since 1967, presenting a very general theory of modal analysis based on the condition contained in (7). This paper by Masterman and Clarricoats [45] is a particularly elegant formulation incorporating both matrix notation and an arbitrary source. The reader interested in furthering his knowledge of modal analysis should consult this paper as a primary reference.

An alternative convergent approach to the matching of fields on the plane P in modal analysis has recently been proposed by Davies [46]. In this paper, the fields across the discontinuity plane are equated by minimizing the functional

$$F = \int [|(\bar{E}_1 - \bar{E}_2) \times \bar{n}|^2 + Z_0 |(\bar{H}_1 - \bar{H}_2) \times \bar{n}|^2] \, dS \tag{8}$$

where the subscripts 1 and 2 indicate the fields in the two guides. The primary attraction of this formulation is that the method is rigorously proved to be convergent in a well

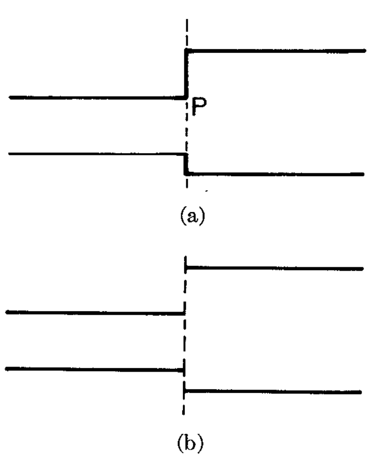


Fig. 7. (a) Transverse discontinuity at junction of two waveguides. (b) Subdivision of discontinuity region at discontinuity plane.

known norm, since it represents a particular case of classical least squares minimization.

Finally, before leaving the subject of modal analysis, it is appropriate to add that the hypothetical waveguide modes defined above are not the only possible functions to use as approximating functions in modal analysis. The work of Burshteyn [47] indicates that another useful set of approximation functions in modal analysis are the fields obtained by short circuiting the discontinuity ends of the equivalent hypothetical waveguides. Furthermore, Knetsch [48] has formulated the modal analysis technique in terms of an arbitrary set of orthogonal functions. Therefore, one should not discount the possibility of using approximating functions other than the waveguide mode functions in modal analysis.

VII. Integral Equation Formulation

The solution of waveguide discontinuity problems using integral equation methods has been established for a long time. A summary of theoretical work based on these methods is provided by Felsen and Kahn [49]. In recent years, the use of the point matching method in the solution of integral equations has become increasingly popular, with a monograph on the subject by Harrington [50] receiving widespread attention. It is regrettable that the almost exclusive use of the point matching technique in this monograph has often led to interchangeable use of the terms *point matching* and *moment method* in microwave literature, contrary to established usage in other fields of science and technology [51].

In a typical integral equation formulation of a waveguide discontinuity problem, an equation of the form

$$\bar{E}_i(r) = \int \overline{\overline{K}}(r,r') \cdot \bar{J}(r')\, dr' \tag{9}$$

is determined where $\bar{E}_i$ is the incident electric field and $\bar{J}$ is the unknown current distribution around the discontinuity. $\bar{J}(r)$ and $\bar{E}_i(r)$ are then approximated by some easily evaluated functions such as polynomials or trigonometric functions and a corresponding approximation of the kernel function $\overline{\overline{K}}$ is made. The resulting matrix equation yields an approximate value for the current distribution on the discontinuity; from this distribution, the value of the total electric field may be determined. The application of this method to the integral equation formulation of waveguide discontinuities has recently been described by Thong [52], Wu and Chow [53], and Chow and Wu [54]. [52] and [53] present a more or less straightforward analysis, while [54] uses an unusual combination of mixed basis functions to approximate the current distribution. In this formulation, the current is written as a sum of incident, scattered, and evanescent terms:

$$\bar{J} = \bar{J}_i + \bar{J}_s + \bar{J}_e \tag{10}$$

with incident and scattered terms being approximated by the corresponding hypothetical waveguide currents. The evanescent current is, however, approximated by pulse functions defined over a finite region surrounding the discontinuity. Thus the method makes good use of the approximating function space: functions with infinite support approximate currents of infinite extent and functions having local support approximate localized currents.

It should be pointed out with regard to the above method that point matching may *not* be regarded as testing with Dirac delta functions as is often asserted, e.g., by Thong [52] and by Chow and Wu [54]. Since the Dirac delta function is not square integrable, it is not an element of the Hilbert space of square-summable functions and may not be used for "testing." In fact, it has been shown by Oden [55], that in addition to the Dirac delta function, finite-dimensional data functions may be defined and that they are equivalent to the conjugate basis functions of interpolation polynomials. Therefore, the point matching procedure is equivalent to testing with the conjugate basis functions to interpolation polynomials. Further, since it is also known that the optimal approximation of a function is made only when the expansion and testing functions in the moment method are conjugates of one another [56], this implies that the most accurate point matching solution will be obtained only when the expansion functions are interpolatory.

Other difficulties are often encountered in the integral equation formulation of iris problems, as evidenced by recent papers by Mittra *et al.* [57] and by Lee *et al.* [58]. Both are concerned with the "relative convergence" phenomenon. Relative convergence is a term used to describe the existence of numerical instability in the approximation of the integral equation by trigonometric expansion functions. Although this phenomenon may be explained analytically, a consequence of the phenomenon is large roundoff error in the solution of an ill-conditioned matrix equation representing the kernel of the integral equation [57]. Thus the cure for relative convergence would appear to be the use of less oscillatory approximation functions, which would result in more stable matrix equations.

With regard to the integral equation solution of waveguide discontinuity problems, it should be mentioned that an attempt has been made by Amitay and Galindo [59] to determine an error estimate for the solution. Sadly, however, the authors report that the usefulness of the error bound is very limited.

VIII. Transverse and Longitudinal Representations

The remaining numerical methods for the solution of waveguide discontinuity problems reduce the three-dimensional boundary value problem in (6) to a two-dimensional one by the method of separation of variables. In the case of a rectangular or circular wave-guiding structure with a uniform discontinuity in the transverse plane, this is accomplished by noting that the transverse field variation is sinusoidal, with the remaining variation satisfying the two-dimensional Helmholtz equation. In the case of a guiding structure with a gradually varying discontinuity in the longitudinal direction, the separation is performed by writing generalized telegraphist's equations for the longitudinal variation of the fields in terms of the waveguide cross-section modes.

An early numerical method for solving discontinuities of uniform cross section in rectangular waveguides was reported by Muilwyk and Davies [60]. They adapted an experimental procedure proposed by Montgomery *et al.* [61] for finding the scattering parameters of a discontinuity by locating the nulls of its standing wave patterns. As indicated by Fig. 8, in a rectangular waveguide containing a discontinuity, a sequence of standing wave patterns may be determined by solving the sequence of field problems defined by taking each of the pairs of planes as locations of field nulls. Muilwyk and Davies solved the associated field problems using the finite-difference method and reported obtaining good accuracy.

A deficiency in the method used by Muilwyk and Davies is that the finite-difference program used was limited to determining only the dominant mode of the structure. Hence, the arbitrarily imposed field null must be taken very close to the discontinuity for a given frequency. Obviously, the use of higher order modes, which would allow the field null closest to the discontinuity to be nonplanar, would result in higher accuracies. Since higher order modes are readily produced by finite-element computer programs, the possibility of using the above formulation in conjunction with the finite-element method is very attractive and awaits future investigation.

Using a similar reasoning, Silvester and Cermak [62] have reported the development of a finite-difference program to solve coaxial line discontinuities in the transverse plane. An advantage of their formulation is that the boundary conditions at the open ends are imposed by boundary relaxation (a procedure available only for TEM structures) and does not introduce the inaccuracy associated with an artificial boundary condition. However, the finite-difference method has been largely superseded by the finite-element method, and boundary relaxation by the method of Silvester and Hsieh [6], so that present day applications of the method should take advantage of the newer more powerful computational techniques.

Another interesting paper making use of this type of transverse discontinuity analysis is by Johns and Beurle [63] who use a network analog method for solving the transverse discontinuity problem. An advantage of their formulation is that the network analog approach facilitates the evaluation of the discontinuity impedance.

Lastly, we turn to the solution of tapered discontinuities in waveguides in the longitudinal direction. The application of numerical methods to this problem has been reported by Davies *et al.* [64]. They use a closed-form solution of the generalized telegraphist's equations, given in terms of the waveguide modes of the varying waveguide cross section, to determine the impedance of the discontinuity. The waveguide modes are computed numerically using the methods outlined in Section III.

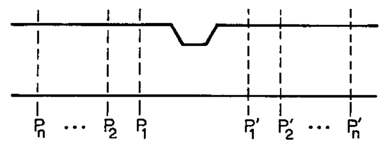


Fig. 8. Sequence of trial null planes for Muilwyk and Davies' analysis of waveguide discontinuities.

IX. Stripline Discontinuities and Planar Circuits

Discontinuities in microstrip, triplate, and other forms of strip transmission structures have recently assumed major importance in microwave engineering as a result of changing technology. These discontinuities are usually modeled by equivalent circuits containing one or a few reactive elements, and occasionally short lengths of idealized TEM line. The reactive element values in such simple models can be found by direct experimental measurement, and considerable amounts of measured data have been gathered. The large variety of possible shapes and materials, however, has rendered mathematical modeling very desirable.

The simplest possible discontinuity in a stripline, the open-circuited end, has probably attracted more attention than any other. Troughton [65] has used experimentally obtained data for multistub filters. Shortly thereafter, Farrar and Adams [66] gave the first computed results. The open-circuited line is modeled similarly to the coaxial cable open circuit of Fig. 1, i.e., the terminal plane is taken at the end of the strip and a single shunting capacitor is assumed to represent the discontinuity adequately. The essential problem of finding the value of this capacitor was solved by Farrar and Adams by finding the capacitance of a finite-length stripline. Such a finite-length section may be modeled by an idealized TEM line connecting two end capacitances; at very low frequencies the finite-line section may be modeled as the shunt combination of three capacitors, the two end-effect capacitors, and the capacitance of the line section itself. By repeating computations for ever longer line lengths and each time subtracting the known electrostatic capacitance of the idealized line, a value for the end capacitance is obtained. The process of taking several lengths of line is necessary to ensure that the charge distributions attributable to the two ends do not interact. To find the capacitance values each time the integral equation governing the charge distribution of the strip

$$V = \int G(P,Q)\rho(Q)\, dQ \qquad (11)$$

is solved numerically using a two-dimensional staircase approximation for the charge density, by a point matching method. In the above, G represents the Green's function appropriate to the problem; P and Q are the observation and source points, respectively, and V is the electric scalar potential. A similar solution, using polynomial approximants and a projective solution technique, was subsequently given by Benedek and Silvester [67].

Whichever way the integral equation is solved, a fundamental difficulty appears. If the finite length of strip is made long enough to ensure that the end effects do not interact, the line capacitance becomes large compared to the end effect capacitance, so that numerical solutions of extremely high accuracy are required to prevent disastrous loss of significant figures when the static line capacitance is subtracted. This difficulty can be circumvented entirely by formulating another integral equation in which the unknown is not the total charge on the strip, but the excess charge attributable to the end effect.

This integral equation is identical in form to the one above, but the Green's function is more complicated. Such a formulation was used by Benedek and Silvester [68] to find equivalent circuits for line gaps and step width changes, as well as for T junctions and crossings [69].

Since there is no way of solving the integral equation for charge distribution analytically, a variety of numerical methods have been employed. In addition to the methods already mentioned, finite-difference techniques and subregion matching methods have been employed [70]–[72]. Another possibility is that of using double Fourier transforms on the integral equation and finding projective solutions in the transform domain. This approach has been pursued by Itoh *et al.* [73].

Although TEM approximations usually lead to acceptable results at low frequencies, striplines containing several dielectric media necessarily propagate hybrid waves. When electrostatic solutions do not suffice, more accurate representations of a microstrip discontinuity may be made by allowing for inductive effects. A typical discontinuity will then be modeled as the combination of its static capacitance and its high-frequency asymptotic inductance. The current distribution in a conductor of arbitrary shape is described [74] by an integral equation similar to that governing the electrostatic charge distribution, but involving the vector current density, instead of the scalar charge density. A solution of this problem has recently been given by Gopinath and Silvester [73].

Microwave planar circuits may be regarded as generalized stripline junctions and are analogous to multiport resonant cavities in waveguide technology. They are readily fabricated in the same process as striplines and have therefore recently attracted attention. In analyzing planar circuits, two approaches have been proposed. The first treats the planar circuit as if it were a combination of stripline junctions, while the second makes use of their similarity to resonant cavities.

In the generalized stripline junction formulation of Okoshi and Miyoshi [76], the basic planar circuit field problem is expressed in terms of an integral equation that is solved numerically. This yields a description of the multiport network in terms of port impedance matrices for each frequency of interest. A second stripline junction formulation for planar circuits has been presented by Bianco and Ridella [77] who performed the analysis using a modified subdomain method. Unfortunately, their work is largely restricted to rectangular networks since they employed trigonometric functions as their basic tools.

More recently, several workers have noted that a resonant-cavity approach permitted a substantial reduction in computing time as well as allowing relatively simple but comprehensible equivalent networks to be drawn. The formulation given by Hsu [78] has produced relatively simple equivalent networks even in the multiport case, while the approach of Silvester [79] has lead directly to port admittance matrices. These formulations are closely related and share the advantage that once the equivalent network, or certain functions characterizing the planar circuit, are known, only very simple arithmetic calculations are required at each frequency of interest.

It will be noted from the above that little if any attempt has so far been made to create synthesis programs for stripline or planar network configurations. However, since all of the analysis methods used rely on representing the network geometry in terms of a finite set of parameters, it seems highly probable that at least some direct synthesis will be possible at an early date.

X. Conclusions

Numerical modeling and analysis methods have permitted a much wider variety of passive microwave devices to be characterized than could possibly be contemplated by experimental methods alone. Most of the numerical techniques currently employed are quite new, and future development can be expected to increase their scope and power. For the present, the majority of modeling techniques are restricted to two-dimensional problems and therefore find direct application only wherever the device to be analyzed is uniform in one direction, so that the inherently three-dimensional field problems associated with the device can be reduced to two-dimensional form by a partial separation of the variables.

The present state of the numerical modeling art can best be summarized by stating that characterizations may be obtained for a large class of devices with accuracies at least adequate for most engineering purposes. However, the device optimization problem, which presumably requires repeated analysis, has as yet hardly been touched. Success in this direction will require considerable future work in two areas. First, further improvement in numerical methods is required to reduce the cost of each individual analysis to a lower level. Second, and perhaps more important, substantial new work is required in turning such geometric variables as shape into numerical quantities, both to facilitate man–machine communication and to permit design optimization algorithms to be constructed in an efficient manner.

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